Epistemology and Ontology

*Bisimulation and duality, a case study.*

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**Abstract**. The main aim of this work is to depict the interconnection of the most relvant formal concepts of modal logic and category theory, i.e., bisimulation and duality, arising from the mathematical analysis of physical processes by showing their relevance with respect to some foundational issues related to the actual ontological debates. Current foundamental physics concerns the non-linear thermodynamics of the quantum field, whose range is made of far-from-equilibrium systems and whose basic mechanism of symmetries (patterns) formation supposes the spontaneous breaking of symmetries (SBS). SBS implies that such systems reach unpredictable states. Thus, evolutive and/or far-from-equilibrium systems are to be conceived primarily as *processes* and just in a secondary way as (“emergent”) objects, for the information they display is always *incomplete* with respect to their evolution (from a fixed state). Formally, this is due to their *non-linear* mathematical behaviour. This make a question about the ontology of such systems, given that the actual most widespread ontologies conceive existent entities just as objects (actualist ontologies).

It is claimed that the fundamental difference and advantage of category theoretic approach to foundation is that, instead of considering objects and operations for what they “are” – as it is in set theory – it considers them for what they “do”. This constitutes a significative shifting in mathematical philosophy and in foundation of mathematical physics: from a Platonic to an Aristotelian ontology (of mathematics and, then, of physics). Actually, providing a contribution to this very shift is what this paper want to be focused on.

# **Introduction**

The main aim of this work is to argue for the involvement of the most relvant formal concepts of modal logic (ML) and category theory (CT), i.e., bisimulation and duality, in the constitution and formalization of some foundational issues arising from the epistemology of physics and to draw some conclusion about the ontology behind the local pattern formation and the stability of physical processes.

Current foundamental physics concerns the non-linear thermodynamics (NLT) of the quantum field, whose range is made of far-from-equilibrium systems and whose basic mechanism of symmetries (patterns) formation supposes the spontaneous breaking of symmetries (SBS). SBS implies that such systems reach unpredictable states. Thus, evolutive and far-from-equilibrium systems are to be conceived primarily as *processes* and just in a secondary way as objects, for the information they display is always *incomplete* with respect to their evolution [Bailly, Longo (2013)]. Formally, this is due to their *non-linear* mathematical behaviour. This make a question about the ontology of such systems, given that the actual most widespread ontologies conceive existent entities just as (existent) objects (i.e., actualist ontologies).[[1]](#footnote-1)

Being a process, i.e., a non-linear process, means being an incomplete entity with respect to the information displayed at a certain state (or moment) of its evolution; what makes a process an actually existent entity (i.e., an object) is the *stability* it can reach far-from-equilibrium conditions, that is, through the (continuous and mutual) matching with energy and information provided by the external as well as internal environment. In this sense, the process is stable, it is an object, untill the pair *system-environment* plays a *dual* game.

The *first* issue where ML and CT converge is precisely this: the *modal* existence of the process. This is governed by the *duality* between the system and the environment that, in turn, is well mathematically represented through the two opposite categories of *algebra* and *coalgebra*, respectively. The link between the pairs system-environment and algebra-coalgebra may be, roughly, expound as follows: if the system provides the *initial* conditions of the process evolution, the environment – or, in NLT vocabulario, its associated *thermal bath* – provides the *final* conditions, namely, the conditions under which the system is made stable. It is exactly thanks to this stability that it “emerges” an actually existing (physical) entity: an object.

The *second* issue is such an *object emergence*. The correlation system-algebra and environment-coalgebra is all but a subjective choice; it is due to two theorems of CT that provide the existence of two mathematical objects: the *initial algebra* and the *final coalgebra* of an endofuntor.[[2]](#footnote-2) By CT, final and initial *objects* of categories, or *categories* themselves, can be related by a *third* *category*[[3]](#footnote-3) of *indexing functors*, so to ensure the mapping[[4]](#footnote-4) of all the objects and morphisms of one category into the other (possibly the same). Thus, by the matching[[5]](#footnote-5) between the elements of the pair system-environment (algebra-coalgebra), this third category ensures stability: a new object (a category) emerges to existences. But what makes all this very appealing is that the endofunctor, as a homomorphism (a preserving map) – but differently from an isomorphism – can´t be reversed over infinite structures: in fact, it may be reversed just over adequate restrictions of them, namely, it may be reversed locally, whitout any supposition of infinitary completeness.[[6]](#footnote-6) Thus, by CT it is possible to build up auto-identities (objecthood)[[7]](#footnote-7) without falling under the set theoretic limitation of diagonal functions[[8]](#footnote-8).

## **What an ontology?**

It is claimed that the fundamental difference and advantage of CT approach to foundation with respect of the set theoretic one is that, instead of considering objects and operations for what they “are” – as it is in set theory and set theoretic logic and ontology – it considers them for what they “do”. This, of course, would constitute a significative shifting in mathematical philosophy: precisely, from a Platonic to an Aristotelian ontology (of mathematics). Actually, this very shift is what this project want to be focused on. In fact, the implicit point this investigation is concerned with is how to treat the potential infinite: in the case of the potentially infinite variation of the domain of quantification. The *modalization of the existence* is the very point. Actuality, here as elsewhere, contrasts with potentiality. But the Aristotelian notion of potentiality differs with all the most widespread ontologies such as Platonism,[[9]](#footnote-9) conceptualism and nominalism[[10]](#footnote-10) inasmuch it does not presupposes any actuality.[[11]](#footnote-11) In fact, it is well known that the Platonic presupposition of set theory consists in the fact “that each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite” [Hallett (1984, p. 25)]. Certainty, Aristotle was the strongest foe of actual infinite.

# **Foundational and epistemological premises**

A preliminary step of this paper consists in tracing a brief discussion on foundation of science and ontology, with special attention to some epistemology and the ontology of the current physical picture.

The endorsement of the belief in the reality of some sort of universal entities[[12]](#footnote-12) with respect to our best scientific theories, in particular, Scientific Realism (SR), are articulated in two main dimensions:

* An *ontic* dimension, committed to the mind-independent existence of the world investigated by the sciences and
* A *semantic* dimension for which claims about scientific primitive entities[[13]](#footnote-13) should be construed literally as having truth values.

From these a third,

* *Epistemic*, dimension arises: that those scientific claims constitutes our knowledge of the (external) world.

After the formalization of ontologies [see Cocchiarella (2007)] the main trouble for realists is to argue in favour of the semantic and epistemic dimensions. The debates and discussions are, indeed, mostly affected by the belief that the foundamental basic entities must, by logic, be “objects”, in any interpretation one wants to give to them. Namely, “they must be individuals or discrete; they must perdure; they must not merge (two objects becoming one); they must not multiply spontaneously (one object becoming two)”. But, whatever one may decide for, all these are “substantive metaphysical assumptions” [Dutilh-Novaes (2013)] which constitutes an ontological bias – till the *scientific method*[[14]](#footnote-14) does not provide a foundation, a grounding, a selection over the plousible choices.

Given the (supposed) mathematical nature of reality,[[15]](#footnote-15) the semantic trouble reduces to the truth of the axioms of set theory, as it is the most widely accepted mathematical theory of objects able to *reduce* all mathematics, after the aritmetization of analysis and function theory due to Weiestrass [Fraenkel and Bar-Hillel (1973)]. The trouble, indeed, is even epistemic and concerns the formulation of a *criterion* of truth[[16]](#footnote-16) [see Cellucci (2014)]. Realism is, thus, seemingly unacceptable: realists, who assume truth as *possession of a model*,[[17]](#footnote-17) are also bound to assume that structures are isomorphic (homomorphic up to infinite domains) to the external world.

But, following Cellucci (2014), “this must be demonstrated in some mathematical theory, and ultimately in Zermelo–Fraenkel set theory (ZF), and presupposes that the axioms of ZF be true. But, by Gödel´s Second Incompleteness Theorem, it is impossible to demonstrate by any absolutely reliable means that the axioms of ZF are true”. In fact, from Gödel´s Theorems, we know that the formalist approach (Hilbert) to foundation fails together with the Platoni(sti)c one (Frege)[[18]](#footnote-18): it is not provable the *absolute consistency* of set theory (in the same formal system)[[19]](#footnote-19) and, hence, the truth of ZF system[[20]](#footnote-20). Further, it is not provable even the absolute (viz. independent) existence of sets[[21]](#footnote-21). The axiomatic (deductive) view of proving the realist instance, thus, had failed definitively.

This axiomatic approach to foundation had an analogous (reductionist) counterpart in treating the information in physical evolutive systems. From Descartes, the modern paradigm was, in fact, affected by the view that natural processes are *linear[[22]](#footnote-22)* (and integrable), i.e., represented by *linear equations*. Modern natural science supported such a conception for at those times the only equations one was able to treat mathematically were linear. But, simultanously, it “produced” a matter of physical content as an immediate consequence, as far as linear equations would have then formally represented the modern concept of *causality*[[23]](#footnote-23), under the dependence of that of “analysis of a situation into simple – viz., mutually independent – elements” [Bridgman (1958, pp. 174-175)].

The parallel between formal and physical systems lies, thus, on the following analogy: *all* the information that describes a modern physical and/or formal system is just impicit in the initial conditions and/or in the axioms. For what concerns physical systems in general, this approach embodies two forms of physical reductionism: (i) by reducing the behaviour of complex systems to the linear laws governing the behavior of their elementary constituents; (ii) by reducing the final state of an evolutive process to its start-point state conditions. Anyway, both approaches are clearly based on different interpretations of the *Closed World Thesis* (CWT) that physical and/or formal systems are closed systems.[[24]](#footnote-24)

Nowadays, there is wide agreement on the non-linear (i.e., dissipative and informationally open) and dynamic (state-transitions systems) feature of foundamental physical processes. The physics of such systems abandones the modern paradigm based on linearity. Indeed, the interplay between Quantum Field Theory (QFT), of dissipative systems,[[25]](#footnote-25) and NLT is its core[[26]](#footnote-26). Briefly summarizing, such systems are characterized by a) SBS (spatial and temporal) and b) the *formation of complex structures*, namely, the interacting particles show *long-range correlations* (viz., formal or unifying interactions), i.e., the EPR experiment.[[27]](#footnote-27) Del Giudice et al.(2009) proves and argues that the most basic physical level, the quantum field, is itself constantly unstable (far-from-equilibrium) and that it constitutes the energetic openess condition for any physical system. According to them, the interplay between QFT and NLT lies on the Third Principle of Thermodynamics (TPT). Their argument maz be resumed as follows.

TPT is, actually, the conceptual basis of foundamental physics.[[28]](#footnote-28)

Accordingly, it is impossible for any physical system to dynamically reach a state of no interactions with the environment or a state of no fluctuation of energy.

Consequently, the concept of *inert isolated body* is declined in foundamental physics.

The argument proves that the notion of *mechanic* or absolute *void* results a mere abstraction, say a hypothesis *to save* phenomena, that follows from the modern bias about linearity operated by the abuse of the so called *linearization methods*[[29]](#footnote-29). Clearly, this account contrasts with any modern and any reductionist approach to physics and, therefore, with CWT.

## **The focus: finitariety**

The range of NLT of the quantum field is made of far-from-equilibrium systems, and its basic mechanism of symmetries (patterns) formation supposes SBS. SBS mechanism is about “local” symmetries (or reversible maps), and this leads to a local foundation of patterns formation. The aim is to investigate how bisimulation and duality interact in the formalization of such a process of local pattern fomation. On the one hand, bisimulation is a relation of *equivalence* between modal models {\itshape weaker} than isomorphism: a form of homomorfism. On the other hand, duality between opposite categories leads to the notion of *duality equivalence* that, in turn, may hold in case of homomorphic dual categories.

The reason for focusing on the interplay between bisimulation and duality is twofold: (1) ML is recognized to be the logic of *coalgebras*: in fact, the notion of *unfolding* that characterizes coalgebraic functors can be seen as an accessibility relation over Kripke models, in case it is a bounded morphism, i.e., a bisimulation [see Blackburn and van Benthem (2007), Venema (2007)]. (2) Coalgebra is something more than just the dual notion of algebra, since it *generalizes* algebra: given that any functor (a morphism of morphisms) can be employed in it.

Coalgebra is, here, the common term. Coalgebra is *finitary* in essence, for a generic functor admits a *final* coalgebra if the coalgebraic category related to that functor has a *final* object[[30]](#footnote-30) [Venema (2007)], namely, there exist a term “of convergence”.[[31]](#footnote-31)

## **The convergence of ML and CT**

On these basis, the philosophical interest for – and the relevance of – CT and ML is to deepen the rationale of their convergence. Such a convergence may be individuated by the following issues:

* The axioms of CT “are formulated purely in terms of the algebraic operations on arrows, without any reference to ‘elements’ of the objects” [Abramsky (2012)].

This makes CT very different from set theory, where objects (sets) have a primacy with respect to any relations (morphisms) that can be defined over them;

* “the most influential ways of thinking about Kripke models” is “to give them a process interpretation”[[32]](#footnote-32) and the most important notion of modal logic is that of *bisimulation* as “a natural notion of process equivalence” [Blackburn and van Benthem (2002, p.14)][[33]](#footnote-33).

In particular, it is worth notice that bisimulation “is a relation between two models, weaker[[34]](#footnote-34) than isomorphism” [Blackburn and van Benthem (2002, p. 2)].

What, then, makes technically converge CT and ML is, definitely, the category of coalgebra, via bisimulation and duality. In fact, according to Venema (2002):

This set-up enables the canonical definition of two notions of equivalence between coalgebras, namely, bisimulation and behavioral equivalence. As we will see as well, the definitions make the concept of a coalgebra very similar to that of an algebra. However, if one makes this connection mathematically precise, it turns out that coalgebras over the base category $C$ are dual to algebras over the opposite category $C^{op}$. [...] Given the nature of coalgebra as a very general model of state-based dynamics, there is a natural place for modal logic as a formalism for reasoning about behavior.

# **The relevance of CT as foundational theory**

Cellucci (2013, pp. 11-13) emphasizes some further limitations of the axiomatic view of formal sciences, i.e., based on higer and general principles, due to its icompatibility with Gödel’s theorems. They are, significative.

* It does not permit to distinguish between mathematical theories in terms of their significance;
* It leads to a fragmentation of the research;
* It cannot explain the succes of the application of mathematics to physics.

Abramsky and Tzevelekos, in the introduction of their (2011), claim that CT is very useful for:

* CT gives a syntax-independent view of the fundamental structures of logic, and opens up new kinds of models and interpretations;
* CT organises mathematics revealing new connections among structures and new structures;
* CT offers new ways of formulating physical theories in a structural form.[[35]](#footnote-35)

As one can see, each point of the first list is matched by each order preserving point in the second list. Accordingly, CT should be able to respond to the failure of the axiomatic and reductionistic approach to foundation. In particular, the latter list is made of interconnected points, and this shows the inner systemic articulation of CT as an organic foundational discipline. CT is an intrinsically semantic theory for a category consists of a structure-preserving collection of arrows (morphism)[[36]](#footnote-36) with a basic operation among them (that of composition, a transitive relation). What makes CT very usefull and adequate for the formalization of dynamic systems is its arrow-theoretic nature that makes it really object independent: in this sense CT may be seen as a process theory. As a consequence, it allows us, indeed, to simultaneously be both properly *specific* and *general*: specific, for mathematical discours is not precise until the structures and which morphisms we are dealing with (which category we are working in) are not specified. At the same time, the fact we are working in a category allows us to be general for any definition or theorem: “by identifying exactly which properties of the ambient category we are using” [Abramsky (2012)]. In particular, definitions reveal a “*normative* force”, given that properties of morphisms, i.e., monomorpfism, epimorphism, isomorphism etc., have sense in any category, i.e., in any mathematical context.

From this observation, the meta-mathematical proper characterization of CT comes quite straighforward. The usage of CT as a suitable meta-language for mathematics, ultimately, is related to the foundamental notion of *functor* $F$ as “morphism between categories”[[37]](#footnote-37), i.e., sending all the objects, arrows, and compositions from a category $C$ into another $D$. In this way, by homomorphism, i.e., a structure preserving map between categories $C$ and $D$[[38]](#footnote-38), not only different mathematical structures may be seen to be of the same sort (category), but even relations among categories themselves may be studied, so to provide a general unification of the many fields without any reductionistic approach[[39]](#footnote-39) that would look to what two structures *are* instead of looking to what two structures *do*.

Further, even new categories may be constued. The main example is that of coalgebras. If the idea of an algebra as a set equipped with some operations is familiar, CT allows us to dualize[[40]](#footnote-40) the usual discussion of algebras in order to obtain a very general notion, that of coalgebras (of an endofunctor[[41]](#footnote-41)). Coalgebras is a new category as far it provides an effective abstraction and a mathematical theory for a central class of computational phenomena[[42]](#footnote-42), among which those of:

* A general notion of observation equivalence between processes;
* A general form of coalgebraic logic (i.e., the logic of coalgebraic structures), which can be seen as a wide-ranging generalization of ML.

All this is emphasized by Abramsky himself bz sazng that “coalgebra provides the basis for a very expressive and flexible theory of discrete, state-based dynamical systems, which seem ripe for much wider application than has been considered thus far” (2012). The relevance of the link between CT and ML, via coalgebra, is then centred on what concerns the modellization of dynamical systems (non-linear processes). Some clarification and some philosophical account these issues is, of course, necessary for the development of this line of research. But, what about the ontological consequences of employng CT as the foundational theory?

## **CT *vs* ZF: an anti-platonistic foundation**

The Platonistic foundation of mathematics has been displayed in many ways. The most common is the second-order foundation of set theory (of mathematics) [see Shapiro (1991)]. A second, position may be that of Quine who, although refusing second-order quantification, considered firts-order set theory a form of Platonism. However, set or model theoretic semantics has to attain at higher-order notions such as “function of functions” or “class of classes” for granting categoricity to its models[[43]](#footnote-43).

This pfenomena are directly linked to our understanding of the notion of set. ZF – and the like – ensures set *total* ordering on foundation axioms such as the *regularity* *axiom*[[44]](#footnote-44). But, on the contrary, this is not the only possible set account. The category **Set** (whose objects are sets and arrows are functions) admits also *non-standard* sets, i.e., those that satisfy the *anti-foundation axiom* (AFA), called non-well-founded (NWF) sets. In NWF-set account set self-belonginess and unbounded chains of set(s) are possible, i.e. $Ω=\{Ω\}$ and “unfolding” this equation $Ω= \left\{\left\{\left\{…\right\}\right\}\right\}$ [Aczel (1988, p. 6)]. This reflexivity means that *only* partial orderings are possible in this framework; this makes NWF-sets more general than standard ZF-sets, for the category of total orderings (**Tos**) is a sub-category of the partial orderings **Pos**[[45]](#footnote-45): contrary to linear orders, that are irreflexive (see footnote 22), partial orders are reflexive and reflexivity makes ordering relation (a set) itself an “object” of the category so to make partial orderings having priorty among ordering relations[[46]](#footnote-46). Then, by using non-well-founded (NWF) sets as the meta-language of logic, such notions as those of “morphism of morphisms” not necessarily are higer-(second-)order notions, contrary to the standard ZF-set theoretic semantic ones of “function of functions” or of “class of classes”.

As an immediate consequence, in CT or NWF frameworks the meta-language of logical and mathematical theories does not imply necessarily any quantification over second-order predicates and functions in the object-language in which it may be translated. Notice that in CT it is even possible to formalize the logical quantifiers as many functors (i.e., *adjoints*) of an algebraic structure over the power-set of a given set, “so to extend to an algebraic form the usual Tarski model-theoretic semantics for first-order logic” [Abramsky and Tzevelekos (2011, p. 45)].

To gloss the theoretic power of NWF-sets and their nexus with ML *via* coalgebras, let me end this section with the following quotation from Blakburn et al (2002, p. 48):

Non-wellfounded sets and many other notions, such as automata and labeled transition systems, have been brought together under the umbrella of co-algebras [...] which form a natural and elegant way to model state-based dynamic systems. Since it was discovered that modal logic is as closely related to co-algebras as equational logic is to algebras, there has been a wealth of results reporting on this connection.

# **ML and bisimulation**

Let turn, now to ML and its basic notion. Propositional ML has a twofold nature with respect to classical predicate logic: it could be regarded as fragments of first or second-order predicate logics, as far modal operators perform quantification without making use of explicit variables and binding. “This idea, when expressed mathematically, has turned out to be the most significant milestone in the history of modal logic” [Blackburn et al (2007, p. xiii)]. The crucial idea is just that necessity becomes linked to the universal quantifier and possibility to the existential one, given their standard meanings with respect to words/points. The “creative difference”[[47]](#footnote-47) is that ML quantification is bounded to “relevant” or “accessible” context lying beyond the actual one: the universes (of worlds/points) “are structured” (the access to them is *mediated*). Two are the consequences, the second of which is relevant for the development of this work. According to Blackburn et al (2007, p. xiii):

1. A number of properties of modal logics follows at once from those of their classical quantificational counterparts;
2. The fragments of classical logic that modal operators correspond to typically have less expressive power than full first-order predicate logic and this results in many new properties. For example, the semantic invariances between models appropriate for modal expressive power are not those of classical logic, but rather turn out to be various forms of *bisimulation*, which preserve *local* properties of worlds and their transition patterns.

These have been and continues to be deepened by modal logicians by discovering and proving many new formal results. Nonetheless, the philosophy behind the employment of bisimulation and related results to ontology has not been yet deepened. A philosophical interpretation of bisimulation applied to the epistemology of non-linear system is provided in the conclusion. In the next subsection I give just a first glance to the connection between bisimulation and coalgebras.

## **5.1. Bisimulation and coalgebra**

Bisimulation, as far as it preserves local properties, is weaker than isomorphism. This phenomenon may be made clear comparing ML and first-order logic in what follows:

Generally speaking, bisimulation plays the same role for modal logic that potential isomorphism[[48]](#footnote-48) does for first-order logic. This can even be made precise in the following sense. To each first-order model $M$ we can associate a modal model whose points are the variable assignments into $M$, and whose accessibility relations are changes from one assignment $g$ to another $g(x := d)$ that resets the value for the variable $x$ to the object $d\in M$. Then two models $M$ and $N$ have a potential isomorphism between them iff their associated modal models are bisimilar. Blackburn and van Benthem (2007, p. 22).

I.e., bisimulation is a homomorphism.[[49]](#footnote-49) And this issue is very relvant since it has been provided an algebraic semantics for ML.[[50]](#footnote-50) The crucial observation in algebraic theory of ML is that standard algebraic constructions correspond to well-known operations on Kripke frames. Indeed, this correspondence can be made precise in the form of categorical duality algebra-coalgebra, which may serve to explain much of the interaction between modal logic and universal algebra. Actually, the coalgebraic perspective on ML is (may be) very fruitful with respect to dynamic systems; Venema (2007, p. 332) comments such an issue:

Coalgebras are simple but foundamental mathematical structures that capture the essence of dynamic or evolving systems. The theory of universal coalgebra seeks to provide a general framework for the study of notions related to (possibly infinite) behavior such as invariance, and observational indistinguishability. When it comes to modal logic, an important difference with the algebraic perspective is that coalgebras generalize rather than dualize the model theory of modal logic. Many familiar notions and constructions, such as bisimulations and bounded morphisms, have analogues in other fields, and find their natural place at the level of coalgebra. Perhaps even more important is the realization that one may generalize the concept of modal logic from Kripke frames to arbitrary coalgebras. In fact, the link between (these generalizations of) modal logic and coalgebra is so tight, that one may even claim that modal logic is the natural logic for coalgebras — just like equational logic is that for algebra.

What makes coalgebras at the same time so relevant for ML and so different from algebras comes from their very definition. In fact, whereas the algebraic operations (morphisms) adopted in the definition of algebras are ways to construct complex objects out of simple ones, coalgebraic operations, going out of the carrier set, should be seen as ways to *unfold* or *observe* objects.[[51]](#footnote-51) Thus, “frames and models are in fact coalgebras in disguise [Venema (2007, p. 391)], for any binary relation $R⊆S×S$ is the function $R\left[⋅\right] :S⟶℘(S)$ mapping a point $s$ to the collection $R[s]$ of its successors.

The idea behind bisimulation is, intuitively, that two states of a system are similar if we cannot distinguish them untill they display the same behavior: in order to prove the identity of two states in a Kripke model/system, it suffices to show that they are linked by some bisimulation.[[52]](#footnote-52) Further, by coinduction,[[53]](#footnote-53) since the class of NWF-sets is the *final* coalgebra of the power-set functor[[54]](#footnote-54), bisimilarity may serve as a notion of identity between “evolving” sets.

Unfortunately, not every functor admits final coalgebras; i.e., **Set**-endofunctors involving the power-set functor; in particular, there is no final Kripke frame or model. Nonetheless, it is a *Fact* that every *small* set functor[[55]](#footnote-55) admits a final coalgebra [see Venema (2007, pp. 396-397)][[56]](#footnote-56). The notion of smallness is easily seen to be equivalent to the instantiation in **Set** of the more general notion of *accessibility* – that characterizes Kripke relational semantics – and it is also equivalent to the concept of boundedness – that characterizes morphisms as bisimilarities.

# **Digressions**

*Modality and paraconsistency*. The paraconsistent nature of ML is nowadays relevant in the contemporary debate on logics and foundational issues of formal sciences. The link among ML, CT and paraconsistency may be sketched in the following way.

The principle of “iterated modality” and the consequent “stratified” nature of the necessity operator may be seen as an unfolding in NWF-sets in which unbounded chains of set inclusions are allowed. This may leads to an original interpretation of the *paraconsistent negation*. Paraconsistent logics are based on the refusal of the *principle of explosion* – *ex falso quodlibet sequitur*. But they are also based on the principle of the *non-coextensive* character of an affirmation with its negation in contradictory statements[[57]](#footnote-57) [Béziau (2000; 2005)]. In particular, the common defense of self-extensionality is due to the view that logic is reductionable to algebra, i.e., that every connective may be viewed as an algebraic operation. Thus, the possibility for an original version of the constructive (i.e., finitary) use of the contradiction, typical of the paraconsistent logics, may be founded with the use of coalgebraic semantics.

Indeed, because of the nested character of the necessity operator, while the negation of $p$, contradicting $p$, negates $p$ at its proper necessity level, $p$ potentially includes all the other propositions not yet unfolded by the iterated modality procedure. But being the unfolding a partially ordered relation, the paraconsitent negation shows a coalgebraic behaviour. Anyway, the contradiction does not propagate itself to the lower unfolded levels of the unfolding of the modal model (NWS-set) tree, given that no total set ordering is allowed in NWF-sets coalgebraic logic.

In other terms, the information (truth) is not conserved between the actual world (the root of the tree) and *all* the consequent valuation points, and this show an unpredicable behaviour. In fact, at each level of the unraveling procedure the actual information *increases*, since a new structure “emerges” in an unpredictable way from the root. Such an “emergence”, however, has the cost of a *decrease* of the potential information included in the precedent unfolded nodes, since, at each of them, the procedure select one only of the two possible branches $<1, 0>$ that were available.[[58]](#footnote-58)

*SBS and partial ordering*. Any quantum field is far-from-equilibium and, as a field of reals, it is a continuum. It is a continuum, of course, but in a *virtual* sense, contrary indeed to the set theoretic definition. Take the quantum field to constitute the universal domain of quantification. In this sense, a symmetry is defined over it, i.e., every two of its elements are isomorphic with respect to being reals. When a spontaneus breaking of symmetries happens, such a field is broken. But, nonetheless, the field is not broken in all of its possible subsets, following the standard notion of powerset. The unfolding of a set (i.e., NWF-sets) plays here a central role, in order to understand how continuum is built up from a denumerably infinite set. NWF-set theory allows, for the anti-foundation axiom, self-memberships. As a first consequences, infinite (unbounded) descendent chains of self-membersed sets exist even if it has a finite graph. This reflexivity means that no total ordering of sets exists, that is, not all sets are comparable (i.e. Axiom of Choice is not valid, at least if taken as its equivalent version of *comparability*), according to the subset ordering relation.[[59]](#footnote-59) Unfolding a set, thus, means that *different inclusion* paths of subsets of a given set (the root) are allowed to be ordered with respect to their direct antecedents and/or successors but not with respect to the different branches of the inclusion tree originating from the root.

With this in mind, it is possible to understand that if the quantum field works as the root, it works as a universal *class* of ZF, but being itself a *set* (contrary to FZ). The foundamental difference among NWF-sets and ZF-sets is that among NWF-sets no total ordering is possible and, then, *not all* infinite set-elements of the power-set *do exist*. Hence, the *continuum is finitary virtual*: indeterminacy is justified and bisimulation, in this context, performs the role of equivalence class substitution between non-extensionally identical NWF-sets, allowing the constitution of new, *local*, symmetries.

*Sability and duality*. Looking at the mechanism that allows any system to maintain its own stability (far-from-equilibrium) over time. In fact, each system is able to maintain stability only if the interaction with its own termal bath (the quantum field) is essentially dual with respect to homomorfism. The stability of a non-linear syistem is in fact justified untill the algebra by which the system is described and the coalgebra by which the quantum field (the thermal bath) in which any system is embedded are dually homomorphic. In this case, thus, any non-linear system and any thermal bath (proper of the system) are said to be *dually equivalent* for the *contravariant* application of the same homomorphic functor.

# **Conclusion**

Let me draw some conclusions. Recall that:

* CT *generalizes* set theory, being not object dependent;

This means that the employment of CT leads to a non-reductionistic way to provide a foundation in formal sciences, i.e., one that does not reduce a system/object into one another but that emphasyzes the structural identities through morphisms. In particular, the notion of duality plays a very relevant role for what concerns analogies and generalization: the category of coalgebra of an endofunctor is, in fact, dually equivalent to the algebra of the controvariant endofunctor.

* NWF-sets *generalize* ZF-sets, and the like, with respect to CT (i.e., **Set**).

This happens for reflexivity and partiality have primacy to totality within the categorical analysis of sets. Partial functions and relations seem, thus, to be very foundative in this context.

* Coalgebra *generalizes* algebra.

For any functor may be employed. This provides a link between CT and ML. The notions of unfolding, that characterizes coalgebraic functors, can be seen as an accessibility relation over Kripke models just in case it is a bounded morphism (bisimulation), such as the finite power-set functor.

The main aim of this work was, then, to shed light on the philosophical relevance of CT and ML in the formalization and comprehension of foundational issues. These, in particular, comes to be of great interest for the formal ontology of processes.

It is a fact, the mainstream debate concerning philosophy of mathematics and the associated formal ontologies is actually devided between only two main position: Platonism and nominalism. By this work it has been emphatized a third option that assimilates, in a new synthesis, and developes the main claims of both: on the one hand, Platonism assumes that numbers and properties have a form of (independent) reality, that second-order logic is the logic of foundation since second-order theories are categorical (infinitistic bias); on the other hand, nominalism assumes that the world is made by/of bare individuals, that first-order logic is the only right (?!) logic for the technical results reached in such a logic: completeness and compactness (finitistic bias). Both position bring some technical difficulties: Platonism is dealt with foundations; nominalism is dealt with Löwenheim-Skolem theorem. Both due to Gödel´s results.

By this work it is argued that from the physical analysis of NLT it is possible to take for good what Platonism and nominalism take for bad of each other, respectively. The ML notion of *bisimulation* and the NWF-set notion of *unfolding* toghether with the CT notion of *duality* between algebra and coalgebra may lead to a *finitiary* foundation of categoricity (individuation of the model for a theory). Simultanously, it is maintained what Platonism and nominalism take for good respectively: on the one hand CT, as well as Platonism, does not consider the existence of bare objects as primary; i.e., objects are identiy morphisms; on the other hand CT, following nominalism, *per sé* does not involve second-order quantification, i.e. adjoints that are functors, namely arrows by definition.

In this sense this third (Aristotelian) stance may arise as a finitary (process) ontology.

# **References**

ACZEL P. (1988). Non-Wellfounded Sets, CLSI Lecture Notes.

ABRAMSKY S. (2012). Logic and categories as tools for building theories. https://arxiv.org/abs/1201.5342.

ABRAMSKY S., TZEVELEKOS N. (2011). Introduction to categories and categorical logic. https://arxiv.org/abs/1102.1313.

BAILLY F., LONGO G., (2013). Mathematics and the Natural Sciences, Imperial College Press.

BELL J.S. (1964). On the Einstein–Podolsky–Rosen paradox. Physics 1:195-290.

BÉZIAU J.-Y. (2000). What is a paraconsistent logic? In: D. Batens et al. (Eds.), Frontiers of paraconsistent logic. Research Studies Press, pp. 95-111.

\_\_\_\_\_\_\_\_. (2005). Paraconsistent logic from a modal viewpoint. Journal of applied logic 3:7-14.

BLACKBURN P., DE RIJKE M., VENEMA Y. (2002). Modal logic. Cambridge UP.

BLACKBURN P., van BENTHEM A. (2007). Modal logic: a semantic perspective. In: Blackburn et al (2007, pp. 1-84).

BLACKBURN P., van BENTHEM A., WOLTER F. (Eds.) (2007). Handbook of modal logic. Elsevier.

BOHM D., (1951). The paradox of Einstein, Rosen and Podolsky. In: Quantum Theory, Dover Publications, pp. 611–623.

BRIDGMAN P.W. (1958). The logic of modern physics. The Macmillan Company.

CELLUCCI C. (2013). Rethinking Logic. Springer.

\_\_\_\_\_\_\_\_. (2014). Cellucci C., (2014). Knowledge, truth and plausibility. Axiomathes 24:517-532.

COCCHIARELLA N.B. (2001). Logic and ontology. Axiomathes, 12:117-150. Springer.

\_\_\_\_\_\_\_\_. (2007). Formal ontology and conceptual realism. Springer.

DA COSTA N.C., ALVES E.H. (1977). Semantical analysis of the calculi $C\_n$. Notre Dame Journal of Formal Logic 18:621-630.

DEL GIUDICE E., PULSELLI R., TIEZZI E. (2009). Thermodynamics of irreversible processes and quantum field theory: an interplay for understanding of ecosystem dynamics. Ecological Modelling 220:1874-1879.

DUTILH-NOVAES C. (2012). The undergeneration of permutation invariance as a criterion of logicality. Erkenntnis (forthcoming).

EINSTEIN A., PODOLSKI B., ROSEN N. (1935). Can quantum mechanical description of physical reality be considered complete? Physical Review 41:777-780.

FRAENKEL A.A., BAR-HILLEL Y., LEVY A. (1973). Foundation of set theory. Elsevier.

GARSON J.W. (2001). Quantification in modal logic. In: D. Gabbay, F. Guenthner (Eds.), Handbook of Philosophical Logic. Vol. III, 267-324. Springer.

GORANKO V., OTTO M. (2007). Model theory of modal logic. In: Blackburn et al (2007, pp. 252-331).

HALLETT M. (1984). Cantorian set theory and limitation of sizes. Clarendo Press.

HORGAN J. (2017). Polymath Stephen Wolfram Defends His Computational Theory of Everything. Scientific American, March 5, 2017. Online at: https://blogs.scientificamerican.com/cross-check/polymath-stephen-wolfram-defends-his-computational-theory-of-everything/

LYRE H. (2004). Holism and Structuralism in U(1) Gauge Theory. *Studies in History and Philosophy of Modern Physics* 35(4).

NERST W. (1969). The new heat theorem. Dover Publications.

SHAPIRO S. (1991). Foundations without foundationalism: a case for second-order logic. Oxford UP.

VENEMA Y. (2007). Algebras and co-algebras. In: Blackburn et al (2007, pp. 331-426).

1. Think of the most commonly used semantics, i.e., set theoretic semantics. In general, non-actualist ontologies differs only for they employ *partial* (*denoting* and/or *interpretation*) functions. This allow them to speak of non-existent objects inasmuch not all terms denotes. The logical context that derives is that of *free logics*, i.e., free of *existential presuppositions*. [↑](#footnote-ref-1)
2. A particular homomorphism, one from and to one and the same category. [↑](#footnote-ref-2)
3. Notice, this is exactly what the quantum mechanics (QM) cannot do. In fact, QM does not have the formal resources to unify two Hilbert spaces, as it happens here by this third category. In this sense, QM and Quantum Field Theory (QFT) are not antagonist. [↑](#footnote-ref-3)
4. Via a *diagonal* functor. [↑](#footnote-ref-4)
5. This matching has as its (natural) interpretation in QFT the *doubling degree of freedom* by which the degree of freedom (dimensions) of the Hilbert space that describes the system. [↑](#footnote-ref-5)
6. For reason that will be clear in next sections, in CT there is no need to suppose the power set of the domain over which the endofunctor is defined. Thus, avoiding any infinitary supposition. For this reason In the present project I speak of *finitariety* as the mathematical feature of locality. [↑](#footnote-ref-6)
7. Objects, in CT, are secondary (“emergent”) entities, inasmuch they are defined as reflexive arrows. [↑](#footnote-ref-7)
8. A diagonal function is a caracteristic function that, for construction (definition), does not belong (as an object) to the set of function it diagonalizes. Cantor´s theorem (the reals are, so to speak, *infinitely more* than natural numbers) is based on this very functions. [↑](#footnote-ref-8)
9. I´m distinguishing within realism between (at least) Platonism and natural realism. Atomism, in matematical contexts, is quite uninteresting for it is formally equivalent to nominalism. [↑](#footnote-ref-9)
10. See, for a precise and formal account of such ontologies, Cocchiarella (2007, 2001). [↑](#footnote-ref-10)
11. See Garson (2001) for what concerns the objectual and intensional (conceptual) in terpretation of quantified ML (QML). In particular, the main reason for the conceptual interpretation of physical reality is that “things, since they change, cannot be identified with term extension. Instead, ... they correspond to term intensions or individual concepts.” (p. 281). But, there are al lot of problem with this. The main, Garson stresses, is that given that individual concepts are functions fron indices or possible worlds to objects, quantifying over them means that “we interpret the domain of any quantifier as a set of all functions” and, then, “we run the risk that the language will have the expressive power of second-order arithmentic, with the result that Gödel's Theorems applies.” (p. 282). [↑](#footnote-ref-11)
12. I.e., properties, classes, natural kinds, etc... [↑](#footnote-ref-12)
13. I.e., events or processes or properties and relations. [↑](#footnote-ref-13)
14. I.e., empirical. [↑](#footnote-ref-14)
15. This supposition induced the modern “Copernican revolution” in philosophy and metaphysics, started by Descartes and ended by Kant and passed through the Galileian-Newtonian revolution in natural sciences. “Induced”, since the foundation of (mathematical) information does not actually need an (transcendental) subject. For three reasons: (i) necessity of mathemathics is no longer synthetic, after Quine's *Two Dogmas*, (ii) nor analytic (if any), after Frege and Russell´s paradox, (iii) nor *a priori*, after Gödel. Information, and hence, mathematical information included, does not presupposes any notion of mind or of (transcendental) subject (see even a very interesting interview of S. Wolfram to Scientific American [Horgan (2017)]). [↑](#footnote-ref-15)
16. A criterion of truth differs from the definition of truth for, while the definition allows us to know under which conditions, if satisfied, a statement is true, it allows to know when a statement is true, namely when those conditions are (or not) satisfied. Thus, the criterion is able to select and fix the model that is used to to satisfy such conditions. [↑](#footnote-ref-16)
17. According to the concept of truth as possession of a model, to know that a sentence is true, one must show that there exists a structure such that every infinite sequence of objects in the structure satisfies the sentence. [↑](#footnote-ref-17)
18. But even the intuitionistc one (Brower). [↑](#footnote-ref-18)
19. By finitary (denumerable or effective) procedures. [↑](#footnote-ref-19)
20. For the formalistic principle: consistency (coherence) implies truth (satisfaction). [↑](#footnote-ref-20)
21. For even the logicist principle – truth implies existence – fails. [↑](#footnote-ref-21)
22. *Linearity* is a property of order, i.e., linear orders. Namely, an order is linear when *all* the member of a given set may be ordered by the given order; or, equivalently, when it is *total* (satisfies trichotomy) and transitive [Enderton (1977)]. Where totality as the specific property of linearity means exactly tha *any/all* elements of the given set are in this order. Notice that linear orders are irreflexive, i.e., it is not the case that any object is in such a given order with itself. Partial orders are in opposition with linear ones, for they are reflexive and not total, but transitive and antisimmetric. These notions will be usefull later on in Sect. 4. [↑](#footnote-ref-22)
23. *Material* and *efficient* causality (*initial* conditions). Forms (essences), as well as final causes (terminal conditions) – whatever intended – were intentionally put aside (Descartes, Newton) or taken back as just (transcendental) epistemic (intentional) forms or concepts (Kant). [↑](#footnote-ref-23)
24. It supposes the axiomatic possession of a model as categorical and not hypotetical. If we conceive mathematical theory as *closed system*, that is, a system based on primitive truths that are given once for all and cannot change, and whose development consists entirely in deducing conclusions from them, i.e., the whole of a mathematical theory is contained in its primitive truths, then, by Gödel’s first incompleteness theorem – for any consistent, sufficiently strong, deductive theory T, there is a sentence G (the Gödelian sentence) of T which is true but cannot be deduced from the axioms of T – “mathematical theories cannot be closed systems. This also implies that Kant’s closed world view of science, considered in Chapter 9, is unjustified” [Cellucci (2013, p. 218)]. [↑](#footnote-ref-24)
25. Say that a system is *dissipative* if, and only if (iff) it operates far from thermodynamic equilibrium. [↑](#footnote-ref-25)
26. Better to say, “a thermodynamic interpretation of QFT”. [↑](#footnote-ref-26)
27. On the completeness of quantum physics see Einstein et al. (1935) and Bohm (1951). Bell inequalities [Bell (1964)] shows that the property of separability does not hold: once having interacted, two quantum objects remain for certain measurements a single object. See Lyre (2004). [↑](#footnote-ref-27)
28. The entropy of a system approaches a constant value as the temperature approaches zero. A classical formulation by Nernst (1969) (actually a consequence of it) is: it is impossible for any process, no matter how idealized, to reduce the entropy of a system to its absolute-zero value in a finite number of operations. [↑](#footnote-ref-28)
29. Methods that tended to favor the formulation of phisical processes in terms of linear differential equations. For instance, the *asyntotic* condition by which the distance between any two system tends toward infinite serves to ensure that any energetic (and informational, in classical physics) exchange between those systems tends toward zero, so to isolate them. Notice, Fourier’s physical and mathematical researches were characterized by such methods. This, obviously, directly affects the very nature of such processes for which we look for the prediction of their evolution (i.e., final states). [↑](#footnote-ref-29)
30. That is, a coalgebra, say $Z$, such that from every coalgebra, saz $A$, in the category, there is a unique *homomorphism* from $A$ to $Z$. [↑](#footnote-ref-30)
31. The finitary character of bisimulation and duality has to be the core of analogous philosophical investigation that would shed light on mathematical features of local pattern formations as well as, further, of ontological and semantical issues about SR, i.e., realist foundations and truth as correspondence and similar. [↑](#footnote-ref-31)
32. Which means that we view models as collections of computational states, and the binary relations as computational actions that transform one state into another. [↑](#footnote-ref-32)
33. That makes two models modally identical. [↑](#footnote-ref-33)
34. A simple homomorfism between infinite models. [↑](#footnote-ref-34)
35. They continue: “There have inter alia been some striking recent applications to quantum information and computation”. [↑](#footnote-ref-35)
36. Any object of the category may be seen as a reflexive arrow. [↑](#footnote-ref-36)
37. That differs from that of morphism between objects (of a category) for it is defined in a twofold way: one that concern its application to the objects (identities) and one that concerns its application to the morphism of its domain category. Further, it has to satisfy the *functoriality condition* that *composition* and *identities* are to be *preserved*. [↑](#footnote-ref-37)
38. This corresponds in set theory to a finite (local) “bijective mapping” between set elements, as composed by suitable finite “injective” and “surjective” functions. [↑](#footnote-ref-38)
39. I.e., one that attempts to reduce the object of a theory to the object of the more foundamental one. [↑](#footnote-ref-39)
40. A category $C$ is dual to another $D$ iff any arrow between any pair of objects of $C$ is inverted in $D$. In this case, $D :=C^{op}$. [↑](#footnote-ref-40)
41. A homomorphism from a category $C$ to itself. [↑](#footnote-ref-41)
42. The fundamental difference between algebras and coalgebras functorially defined, that makes coalgebras a generalization of agebras, consist in the fact that in the category of coalgebras we are dealing with arbitrary set functors, while in the category of algebras we are constrained – for the Fundamental Theorem of Algebra – to deal with functors that can be only polynomials. [↑](#footnote-ref-42)
43. The power of categoricity is weakened by the semantic incompleteness of second-order logic (due to Gödel), i.e., two model for the same second-order theory are always isomorphic but the set of axioms (or rules) for second-order logic is not complete i.e., not fixed once for all; hence, the syntactic resources are not enough for prooving all truths. [↑](#footnote-ref-43)
44. Every non empty set $A$ has an element $b$ such that $b$ and $A$ are disjoint. By this axiom, in ZF it is avoided the case: (i) a set is member of itself; (ii) there exist an infinite descendent succession of sets. That is, it ensures the existence of *Ur-elements* even when they are avoided by the axiomatization. [↑](#footnote-ref-44)
45. Any category is technically a pratial order, being characterized by transitive and reflexive operations (composition and identity). [↑](#footnote-ref-45)
46. Think of that of **Preorder**, i.e., that satisfy just reflexivity (identity) and transitivity (composition). [↑](#footnote-ref-46)
47. With respect to the simple formal analysis of modal operators in terms of quantification and *vice versa*. [↑](#footnote-ref-47)
48. It is an “invariance condition for a weaker notions of structural equivalence. [...] [A] potential isomorphism between two models $M$ and $N$ is a non-empty set $I$ of finite partial isomorphisms satisfying the back-and-forth extension conditions that, whenever $f\in I$ and $d\in M$, then there is an $e\in N$ such that $f∪\left\{\left(d, e\right)\right\}\in I$, and *vice-versa*. Note that isomorphisms induce potential isomorphisms [...]. The converse is not true. [...] It is easy to show that all first-order formulas are invariant for potential isomorphism, but the real match is with a stronger language: two models are potentially isomorphic iff they have the same complete theory in the infinitary first-order logic $L\_{\infty ω}$.” [Blackburn and van Benthem (2007, p. 18)]. [↑](#footnote-ref-48)
49. “Taking an example from coalgebra, the notion of a bounded morphism [i.e., bisimulation] between Kripke models (or frames), becomes much more natural once we understand that it coincides with the natural coalgebraic notion of a homomorphism” [Venema (2007, p. 333)]. [↑](#footnote-ref-49)
50. By Jonsson and Tarski in the 1950s. One advantage of the algebraic semantics over the relational one is that it allows a general completeness result. [↑](#footnote-ref-50)
51. This explains the central role of the notion of behavior in the theory of coalgebras. [↑](#footnote-ref-51)
52. Much weaker, then, extensionality end equivalence class relation (isomorphisms), common to the standard set theoretic or algebraic theoretic logical framework. [↑](#footnote-ref-52)
53. The principle of coinduction, dual to induction, serves both as an important proof tool and as an elegant means of providing definitions. As a definition principle, coinduction is based on the existence of a unique homomorphisms into the final coalgebra of an endofunctor. [↑](#footnote-ref-53)
54. Recall that a functor admits a final coalgebra if the coalgebraic category related to that functor has a final object, that is, a coalgebra $Z$ such that from every coalgebra $A$ in the category, there is a unique homomorphism from $A$ to $Z$. [↑](#footnote-ref-54)
55. “Examples of small functors abound; for instance, whenever we replace, in a Kripke polynomial functor, the power-set functor by a bounded variant such as the finite power-set functor, the result is a small functor. For instance, the finite power-set functor $℘\_{ω}$ is $ω$-small”. [↑](#footnote-ref-55)
56. “As one of the immediate corollaries of this *Fact*, the categories of image finite frames and image finite models, which can be represented as coalgebras for the functor $℘\_{ω}$, and $℘(Prop)×℘\_{ω}$, respectively, have final objects”. [↑](#footnote-ref-56)
57. This property of negation is called self-extensionality: a negation is self-extensional if two equivalent propositions have equivalent negations (and hence, they can be substituted). To say that a logic is self-extensional is to say that replacement holds for it. [↑](#footnote-ref-57)
58. See Da Costa and Alves (1977) for the hierarchy of logical systems, and of the related algebras, proper of Da Costa paraconsistent logic. [↑](#footnote-ref-58)
59. Recall that in order to construct natural numbers from sets, inclusion, as well as membership are even transitive relations. [↑](#footnote-ref-59)